Sets, Relations and Binary Operations

Set
Set is a collection of well defined objects which are distinct from each other. Sets are usually denoted by capital letters $A, B, C, \ldots$ and elements are usually denoted by small letters $a, b, c, \ldots$.

If $a$ is an element of a set $A$, then we write $a \in A$ and say $a$ belongs to $A$ or $a$ is in $A$ or $a$ is a member of $A$. If $a$ does not belong to $A$, we write $a \notin A$.

Standard Notations

- $N$: A set of natural numbers.
- $W$: A set of whole numbers.
- $Z$: A set of integers.
- $Z^+ / Z^-$: A set of all positive/negative integers.
- $Q$: A set of all rational numbers.
- $Q^+ / Q^-$: A set of all positive/negative rational numbers.
- $R$: A set of real numbers.
- $R^+ / R^-$: A set of all positive/negative real numbers.
- $C$: A set of all complex numbers.

Methods for Describing a Set

(i) Roster/Listing Method/Tabular Form In this method, a set is described by listing element, separated by commas, within braces.

e.g. $A = \{a, e, i, o, u\}$
(ii) **Set Builder/Rule Method**  In this method, we write down a property or rule which gives us all the elements of the set by that rule.

e.g. \( A = \{ x : x \text{ is a vowel of English alphabets} \} \)

**Types of Sets**

(i) **Finite Set**  A set containing finite number of elements or no element.

(ii) **Cardinal Number of a Finite Set**  The number of elements in a given finite set is called cardinal number of finite set, denoted by \( n(A) \).

(iii) **Infinite Set**  A set containing infinite number of elements.

(iv) **Empty/Null/Void Set**  A set containing no element, it is denoted by \( \phi \) or \( \{ \} \).

(v) **Singleton Set**  A set containing a single element.

(vi) **Equal Sets**  Two sets \( A \) and \( B \) are said to be equal, if every element of \( A \) is a member of \( B \) and every element of \( B \) is a member of \( A \) and we write \( A = B \).

(vii) **Equivalent Sets**  Two sets are said to be equivalent, if they have same number of elements.

If \( n(A) = n(B) \), then \( A \) and \( B \) are equivalent sets.

(viii) **Subset and Superset**  Let \( A \) and \( B \) be two sets. If every element of \( A \) is an element of \( B \), then \( A \) is called subset of \( B \) and \( B \) is called superset of \( A \).

Written as \( A \subseteq B \) or \( B \supseteq A \)

(ix) **Proper Subset**  If \( A \) is a subset of \( B \) and \( A \neq B \), then \( A \) is called proper subset of \( B \) and we write \( A \subset B \).

(x) **Universal Set**  \( (U) \)  A set consisting of all possible elements which occurs under consideration is called a universal set.

(xi) **Comparable Sets**  Two sets \( A \) and \( B \) are comparable, if \( A \subseteq B \) or \( B \subseteq A \).

(xii) **Non-Comparable Sets**  For two sets \( A \) and \( B \), if neither \( A \subseteq B \) nor \( B \subseteq A \), then \( A \) and \( B \) are called non-comparable sets.

(xiii) **Power Set**  The set formed by all the subsets of a given set \( A \), is called power set of \( A \), denoted by \( P(A) \).

(xiv) **Disjoint Sets**  Two sets \( A \) and \( B \) are called disjoint, if \( A \cap B = \phi \), i.e. they do not have any common element.
Venn Diagram
In a Venn diagram, the universal set is represented by a rectangular region and a set is represented by circle or a closed geometrical figure inside the universal set.

Operations on Sets
1. Union of Sets
The union of two sets \( A \) and \( B \), denoted by \( A \cup B \), is the set of all those elements, each one of which is either in \( A \) or in \( B \) or both in \( A \) and \( B \).

2. Intersection of Sets
The intersection of two sets \( A \) and \( B \), denoted by \( A \cap B \), is the set of all those elements which are common to both \( A \cap B \) and .

If \( A_1, A_2, ..., A_n \) is a finite family of sets, then their intersection is denoted by \( \bigcap_{i=1}^{n} A_i \) or \( A_1 \cap A_2 \cap ... \cap A_n \).

3. Complement of a Set
If \( A \) is a set with \( U \) as universal set, then complement of a set \( A \), denoted by \( A' \) or \( A^c \) is the set \( U - A \).

4. Difference of Sets
For two sets \( A \) and \( B \), the difference \( A - B \) is the set of all those elements of \( A \) which do not belong to \( B \).

5. Symmetric Difference
For two sets \( A \) and \( B \), symmetric difference is the set \( (A - B) \cup (B - A) \) denoted by \( A \Delta B \).
Laws of Algebra of Sets

For three sets $A$, $B$ and $C$

(i) **Idempotent Law**
   (a) $A \cup A = A$
   (b) $A \cap A = A$

(ii) **Identity Law**
    (a) $A \cup \emptyset = A$
    (b) $A \cap U = A$

(iii) **Commutative Law**
    (a) $A \cup B = B \cup A$
    (b) $A \cap B = B \cap A$

(iv) **Associative Law**
    (a) $(A \cup B) \cup C = A \cup (B \cup C)$
    (b) $A \cap (B \cap C) = (A \cap B) \cap C$

(v) **Distributive Law**
    (a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
    (b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(vi) **De-Morgan’s Law**
    (a) $(A \cup B)' = A' \cap B'$
    (b) $(A \cap B)' = A' \cup B'$

(vii) (a) $A - (B \cap C) = (A - B) \cup (A - C)$
      (b) $A - (B \cup C) = (A - B) \cap (A - C)$

(viii) (a) $A - B = A \cap B'$
       (b) $B - A = B \cap A'$
       (c) $A - B = A \Leftrightarrow A \cap B = \emptyset$
       (d) $(A - B) \cup B = A \cup B$
       (e) $(A - B) \cap B = \emptyset$
       (f) $A \cap B \subseteq A$ and $A \cap B \subseteq B$
       (g) $A \cup (A \cap B) = A$
       (h) $A \cap (A \cup B) = A$

(ix) (a) $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$
    (b) $A \cap (B - C) = (A \cap B) - (A \cap C)$
    (c) $A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$
    (d) $(A \cap B) \cup (A - B) = A$
    (e) $A \cup (B - A) = (A \cup B)$

(x) (a) $U' = \emptyset$
    (b) $\emptyset' = U$
    (c) $(A')' = A$
    (d) $A \cap A' = \emptyset$
    (e) $A \cup A' = U$
    (f) $A \subseteq B \Leftrightarrow B' \subseteq A'$
Important Points to be Remembered

(i) Every set is a subset of itself i.e. \( A \subseteq A \), for any set \( A \).
(ii) Empty set \( \emptyset \) is a subset of every set i.e. \( \emptyset \subseteq A \), for any set \( A \).
(iii) For any set \( A \) and its universal set \( U \), \( A \subseteq U \).
(iv) If \( A = \emptyset \), then power set has only one element
\[ n(P(A)) = 1 \]
(v) Power set of any set is always a non-empty set.
(vi) Suppose \( A = \{1, 2\} \), then
\[ P(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, \emptyset\} \]
(a) \( A \in P(A) \)
(b) \( \{A\} \in P(A) \)
(vii) If a set \( A \) has \( n \) elements, then \( P(A) \) or subset of \( A \) has \( 2^n \) elements.
(viii) Equal sets are always equivalent but equivalent sets may not be equal.
(ix) The set \( \emptyset \) is not a null set. It is a set containing one element \( \emptyset \).

Results on Number of Elements in Sets

(i) \( n(A \cup B) = n(A) + n(B) - n(A \cap B) \)
(ii) \( n(A \cup B) = n(A) + n(B) \), if \( A \) and \( B \) are disjoint sets.
(iii) \( n(A - B) = n(A) - n(A \cap B) \)
(iv) \( n(B - A) = n(B) - n(A \cap B) \)
(v) \( n(A \Delta B) = n(A) + n(B) - 2n(A \cap B) \)
(vi) \( n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C) \)
(vii) \( n \) (number of elements in exactly two of the sets \( A, B, C \))
\[ = n(A \cap B) + n(B \cap C) + n(C \cap A) - 3n(A \cap B \cap C) \]
(viii) \( n \) (number of elements in exactly one of the sets \( A, B, C \))
\[ = n(A) + n(B) + n(C) - 2n(A \cap B) - 2n(B \cap C) - 2n(A \cap C) + 3n(A \cap B \cap C) \]
(ix) \( n(A' \cup B') = n(A \cap B') = n(U) - n(A \cap B) \)
(x) \( n(A' \cap B') = n(A \cup B') = n(U) - n(A \cup B) \)

Ordered Pair
An ordered pair consists of two objects or elements in a given fixed order.
Equality of Ordered Pairs Two ordered pairs \((a_1, b_1)\) and \((a_2, b_2)\) are equal, iff \( a_1 = a_2 \) and \( b_1 = b_2 \).
Cartesian Product of Sets
For two sets $A$ and $B$ (non-empty sets), the set of all ordered pairs $(a, b)$ such that $a \in A$ and $b \in B$ is called Cartesian product of the sets $A$ and $B$, denoted by $A \times B$.

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$$

If there are three sets $A$, $B$, $C$ and $a \in A$, $b \in B$ and $c \in C$, then we form an ordered triplet $(a, b, c)$. The set of all ordered triplets $(a, b, c)$ is called the cartesian product of these sets $A$, $B$ and $C$.

i.e. $$A \times B \times C = \{(a, b, c) : a \in A, b \in B, c \in C\}$$

Properties of Cartesian Product
For three sets $A$, $B$ and $C$

(i) $n(A \times B) = n(A) \cdot n(B)$

(ii) $A \times B = \emptyset$, if either $A$ or $B$ is an empty set.

(iii) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(iv) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(v) $A \times (B - C) = (A \times B) - (A \times C)$

(vi) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

(vii) If $A \subseteq B$ and $C \subseteq D$, then $(A \times C) \subseteq (B \times D)$

(viii) If $A \subseteq B$, then $A \times A \subseteq (A \times B) \cap (B \times A)$

(ix) $A \times B = B \times A$ $\iff$ $A = B$

(x) If either $A$ or $B$ is an infinite set, then $A \times B$ is an infinite set.

(xi) $A \times (B' \cup C') = (A \times B) \cup (A \times C)$

(xii) $A \times (B' \cap C') = (A \times B) \cup (A \times C)$

(xiii) If $A$ and $B$ be any two non-empty sets having $n$ elements in common, then $A \times B$ and $B \times A$ have $n^2$ elements in common.

(xiv) If $A \neq B$, then $A \times B \neq B \times A$

(xv) If $A = B$, then $A \times B = B \times A$

(xvi) If $A \subseteq B$, then $A \times C \subseteq B \times C$ for any set $C$.

Relation
If $A$ and $B$ are two non-empty sets, then a relation $R$ from $A$ to $B$ is a subset of $A \times B$.

If $R \subseteq A \times B$ and $(a, b) \in R$, then we say that $a$ is related to $b$ by the relation $R$, written as $aRb$. 
Domain and Range of a Relation

Let \( R \) be a relation from a set \( A \) to set \( B \). Then, set of all first components or coordinates of the ordered pairs belonging to \( R \) is called the domain of \( R \), while the set of all second components or coordinates of the ordered pairs belonging to \( R \) is called the range of \( R \).

Thus, domain of \( R = \{ a : (a, b) \in R \} \) and range of \( R = \{ b : (a, b) \in R \} \)

Types of Relation

(i) **Void Relation**  As \( \phi \subseteq A \times A \), for any set \( A \), so \( \phi \) is a relation on \( A \), called the empty or void relation.

(ii) **Universal Relation**  Since, \( A \times A \subseteq A \times A \), so \( A \times A \) is a relation on \( A \), called the universal relation.

(iii) **Identity Relation**  The relation \( I_A = \{(a, a) : a \in A\} \) is called the identity relation on \( A \).

(iv) **Reflexive Relation**  A relation \( R \) is said to be reflexive relation, if every element of \( A \) is related to itself.

Thus, \( (a, a) \in R, \forall a \in A \Rightarrow R \) is reflexive.

(v) **Symmetric Relation**  A relation \( R \) is said to be symmetric relation, iff

\[
(a, b) \in R \Rightarrow (b, a) \in R, \forall a, b \in A
\]

i.e.

\[
a R b \Rightarrow bRa, \forall a, b \in A
\]

\( \Rightarrow R \) is symmetric.

(vi) **Anti-Symmetric Relation**  A relation \( R \) is said to be anti-symmetric relation, iff

\[
(a, b) \in R \text{ and } (b, a) \in R \Rightarrow a = b, \forall a, b \in A
\]

(vii) **Transitive Relation**  A relation \( R \) is said to be transitive relation, iff \( (a, b) \in R \) and \( (b, c) \in R \)

\( \Rightarrow (a, c) \in R, \forall a, b, c \in A \)

(viii) **Equivalence Relation**  A relation \( R \) is said to be an equivalence relation, if it is simultaneously reflexive, symmetric and transitive on \( A \).

(ix) **Partial Order Relation**  A relation \( R \) is said to be a partial order relation, if it is simultaneously reflexive, symmetric and anti-symmetric on \( A \).

(x) **Total Order Relation**  A relation \( R \) on a set \( A \) is said to be a total order relation on \( A \), if \( R \) is a partial order relation on \( A \).
Inverse Relation
If $A$ and $B$ are two non-empty sets and $R$ be a relation from $A$ to $B$, such that $R = \{(a, b): a \in A, b \in B\}$, then the inverse of $R$, denoted by $R^{-1}$, is a relation from $B$ to $A$ and is defined by

$$R^{-1} = \{(b, a): (a, b) \in R\}$$

Equivalence Classes of an Equivalence Relation
Let $R$ be equivalence relation in $A (\neq \emptyset)$. Let $a \in A$.
Then, the equivalence class of $a$ denoted by $[a]$ or $\{\bar{a}\}$ is defined as the set of all those points of $A$ which are related to $a$ under the relation $R$.

Composition of Relation
Let $R$ and $S$ be two relations from sets $A$ to $B$ and $B$ to $C$ respectively, then we can define relation $S o R$ from $A$ to $C$ such that $(a, c) \in S o R \iff \exists b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$.
This relation $S o R$ is called the composition of $R$ and $S$.
(i) $R o S \neq S o R$
(ii) $(S o R)^{-1} = R^{-1} o S^{-1}$ known as reversal rule.

Congruence Modulo $m$
Let $m$ be an arbitrary but fixed integer. Two integers $a$ and $b$ are said to be congruence modulo $m$, if $a - b$ is divisible by $m$ and we write $a \equiv b \pmod{m}$.
i.e. $a \equiv b \pmod{m} \iff a - b$ is divisible by $m$.

Important Results on Relation
(i) If $R$ and $S$ are two equivalence relations on a set $A$, then $R \cap S$ is also an equivalence relation on $A$.
(ii) The union of two equivalence relations on a set is not necessarily an equivalence relation on the set.
(iii) If $R$ is an equivalence relation on a set $A$, then $R^{-1}$ is also an equivalence relation on $A$.
(iv) If a set $A$ has $n$ elements, then number of reflexive relations from $A$ to $A$ is $2^{n^2}$.
(v) Let $A$ and $B$ be two non-empty finite sets consisting of $m$ and $n$ elements, respectively. Then, $A \times B$ consists of $mn$ ordered pairs. So, the total number of relations from $A$ to $B$ is $2^{mn}$. 
Binary Operations
Let $S$ be a non-empty set. A function $f$ from $S \times S$ to $S$ is called a binary operation on $S$ i.e. $f : S \times S \to S$ is a binary operation on set $S$.

Closure Property
An operation $*$ on a non-empty set $S$ is said to satisfy the closure property, if
$$a \in S, b \in S \Rightarrow a * b \in S, \forall a, b \in S$$
Also, in this case we say that $S$ is closed for $*$. An operation $*$ on a non-empty set $S$, satisfying the closure property is known as a binary operation.

Properties
(i) Generally binary operations are represented by the symbols $*$, $\oplus$, $\ominus$, etc., instead of letters figure etc.
(ii) Addition is a binary operation on each one of the sets $N, Z, Q, R$ and $C$ of natural numbers, integers, rationals, real and complex numbers, respectively. While addition on the set $S$ of all irrationals is not a binary operation.
(iii) Multiplication is a binary operation on each one of the sets $N, Z, Q, R$ and $C$ of natural numbers, integers, rationals, real and complex numbers, respectively. While multiplication on the set $S$ of all irrationals is not a binary operation.
(iv) Subtraction is a binary operation on each one of the sets $Z, Q, R$ and $C$ of integers, rationals, real and complex numbers, respectively. While subtraction on the set of natural numbers is not a binary operation.
(v) Let $S$ be a non-empty set and $P(S)$ be its power set. Then, the union, intersection and difference of sets, on $P(S)$ is a binary operation.
(vi) Division is not a binary operation on any of the sets $N, Z, Q, R$ and $C$. However, it is not a binary operation on the sets of all non-zero rational (real or complex) numbers.
(vii) Exponential operation $(a, b) \rightarrow a^b$ is a binary operation on set $N$ of natural numbers while it is not a binary operation on set $Z$ of integers.
Types of Binary Operations

(i) **Associative Law** A binary operation * on a non-empty set $S$ is said to be associative, if $(a * b) * c = a * (b * c), \forall a, b, c \in S.$

Let $R$ be the set of real numbers, then addition and multiplication on $R$ satisfies the associative law.

(ii) **Commutative Law** A binary operation * on a non-empty set $S$ is said to be commutative, if

$$a * b = b * a, \forall a, b \in S.$$ 

Addition and multiplication are commutative binary operations on $Z$ but subtraction not a commutative binary operation, since

$$2 - 3 \neq 3 - 2.$$ 

Union and intersection are commutative binary operations on the power set $P(S)$ of all subsets of set $S.$ But difference of sets is not a commutative binary operation on $P(S)$.

(iii) **Distributive Law** Let * and o be two binary operations on a non-empty sets. We say that * is distributed over o, if

$$a * (b o c) = (a * b) o (a * c), \forall a, b, c \in S$$

also called (left distribution) and

$$(b o c) * a = (b * a) o (c * a), \forall a, b, c \in S$$

also called (right distribution).

Let $R$ be the set of all real numbers, then multiplication distributes addition on $R.$

Since,

$$a \cdot (b + c) = a \cdot b + a \cdot c, \forall a, b, c \in R.$$ 

(iv) **Identity Element** Let * be a binary operation on a non-empty set $S.$ An element $e \in S,$ if it exist such that

$$a * e = e * a = a, \forall a \in S.$$ 

is called an identity elements of $S,$ with respect to *.

For addition on $R,$ zero is the identity elements in $R.$

Since, $a + 0 = 0 + a = a, \forall a \in R$

For multiplication on $R,$ 1 is the identity element in $R.$

Since, $a \times 1 = 1 \times a = a, \forall a \in R$

Let $P(S)$ be the power set of a non-empty set $S.$ Then, $\emptyset$ is the identity element for union on $P(S)$ as

$$A \cup \emptyset = \emptyset \cup A = A, \forall A \in P(S).$$
Also, S is the identity element for intersection on \( P(S) \).
Since, \( A \cap S = A \cap S = A, \forall A \in P(S) \).
For addition on \( N \) the identity element does not exist. But for multiplication on \( N \) the identity element is 1.

(v) **Inverse of an Element**  Let \( * \) be a binary operation on a non-empty set \( S \) and let \( e \) be the identity element.
Let \( a \in S \) we say that \( a^{-1} \) is invertible, if there exists an element \( b \in S \) such that \( a * b = b * a = e \)
Also, in this case, \( b \) is called the inverse of \( a \) and we write, \( a^{-1} = b \)
Addition on \( N \) has no identity element and accordingly \( N \) has no invertible element.
Multiplication on \( N \) has 1 as the identity element and no element other than 1 is invertible.
Let \( S \) be a finite set containing \( n \) elements.
Then, the total number of binary operations on \( S \) is \( n^{n^2} \).
Let \( S \) be a finite set containing \( n \) elements.
Then, the total number of commutative binary operation on \( S \) is \( \frac{n \cdot (n + 1)}{2} \).